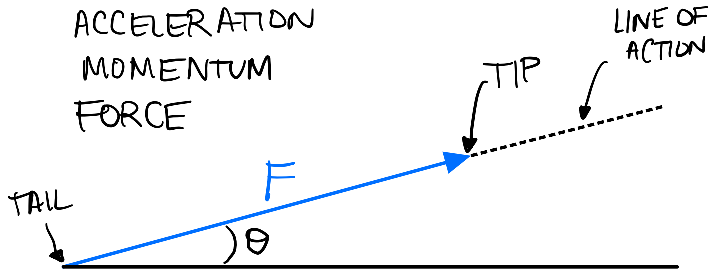
FORCES & OTHER VECTORS

MANY QUANTITIES IN ENGINEERING CAN BE EXPRESSED AS SCALARS OR VECTORS.

SCALAR QUANTITY	VECTOR QUANTITY
HAS ONLY MAGNITUDE	HAS BOTH MAGNITUDE AND DIRECTION
DISTANCE	DISPLACEMENT
SPEED	VELOCITY
MASS	WEIGHT
DENSITY	ACCELERATION
TEMPERATURE	MOMENTUM
ENERGY	FORCE
POWER	
WORK	



STANDARD NOTATION FOR A VECTOR IN PRINTED TEXT WILL OFTEN USE THE VECTOR'S NAME IN BOLD FONT. FOR HANDWRITTEN WORK, WE'LL USE AN ARROW OVER THE VECTOR'S NAME.

$$\mathbf{F} = \vec{F} = \vec{\mathrm{F}} = \text{A VECTOR NAMED F}$$

THE MAGNITUDE OF A VECTOR IS SHOWN GRAPHICALLY BY THE SIZE OF THE ARROW. IN STANDARD NOTATION IN PRINTED TEXT, IT IS GIVEN BY THE VECTOR'S NAME IN ITALICS. SYMBOLICALLY, THE MAGNITUDE OF A VECTOR IS FOUND USING THE SAME NOTATION AS AN ABSOLUTE VALUE. (ABSOLUTE VALUE OF A NUMBER AND MAGNITUDE OF A VECTOR CAN BOTH BE THOUGHT OF AS A DISTANCE FROM THE ORIGIN, SO THIS NOTATION MAKES SENSE.)

$$F = |\mathbf{F}| = \mathrm{F} = |\vec{\mathrm{F}}| = \text{THE MAGNITUDE OF VECTOR } \vec{\mathrm{F}}$$

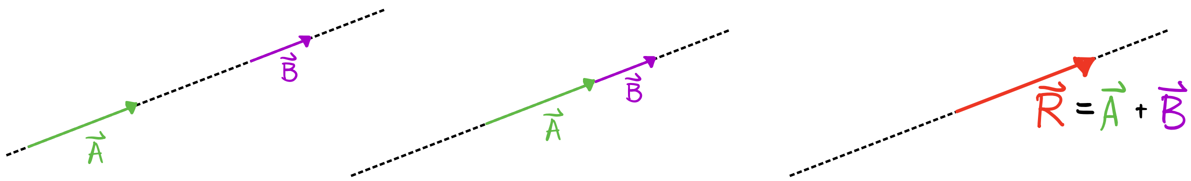
1-DIMENSIONAL VECTORS

THE SIMPLEST VECTOR CALCULATIONS CAN BE DONE WITH 1-D VECTORS, WHICH ARE VECTORS SHARING THE SAME LINE OF ACTION. THESE ARE ALSO CALLED COLLINEAR VECTORS.

VECTOR ADDITION

MULTIPLE VECTORS CAN BE ADDED TOGETHER TO FIND A RESULTANT VECTOR

TO FIND THE RESULTANT VECTOR $\vec{R}$ OF THE TWO COLLINEAR VECTORS $\vec{A}$ AND $\vec{B}$ SHOWN BELOW, WE CAN USE THE TIP-TO-TAIL TECHNIQUE.



TO ADD $\vec{A}$ AND $\vec{B}$:

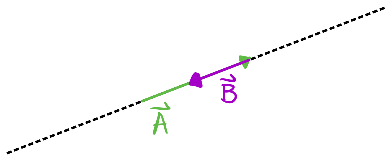
1. SLIDE VECTOR $\vec{B}$ UNTIL ITS TAIL IS AT THE TIP OF $\vec{A}$
2. THE VECTOR FROM THE TAIL OF $\vec{A}$ TO THE TIP OF $\vec{B}$ IS THE RESULTANT $\vec{R}$

VECTOR SUBTRACTION

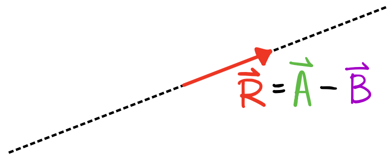
THE EASIEST WAY TO HANDLE VECTOR SUBTRACTION IS TO ADD THE NEGATIVE OF THE VECTOR YOU WANT TO SUBTRACT. THIS WAY, YOU CAN STILL USE THE TIP-TO-TAIL TECHNIQUE.

TO SUBTRACT $\vec{B}$ FROM $\vec{A}$:

1. MULTIPLY $\vec{B}$ BY -1 TO FLIP ITS DIRECTION 180°.



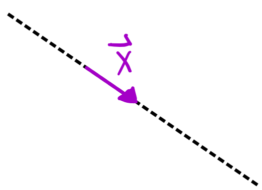
2. SLIDE VECTOR $\vec{B}$ UNTIL ITS TAIL IS AT THE TIP OF $\vec{A}$



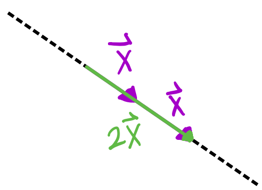
3. THE VECTOR FROM THE TAIL OF $\vec{A}$ TO THE TIP OF $\vec{B}$ IS THE RESULTANT $\vec{R}$

VECTOR MULTIPLICATION BY A SCALAR

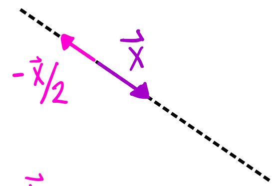
MULTIPLYING OR DIVIDING A VECTOR BY A SCALAR CHANGES THE VECTOR'S MAGNITUDE, BUT RETAINS THE ORIGINAL LINE OF ACTION. MULTIPLYING A VECTOR BY A NEGATIVE SCALAR FLIPS THE DIRECTION OF THE VECTOR (ROTATES IT 180°).



LET'S MULTIPLY AND DIVIDE $\vec{X}$ BY SCALARS.



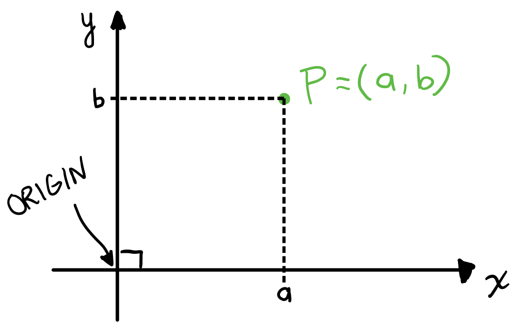
$2\vec{X}$ IS TWICE THE MAGNITUDE OF $\vec{X}$, BUT HAS THE SAME DIRECTION.



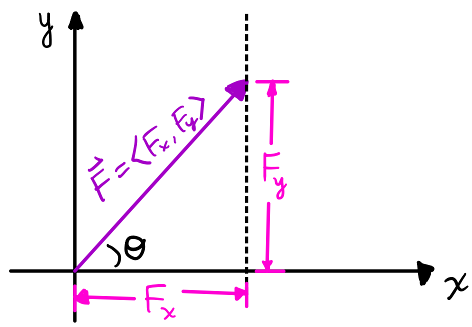
$-\vec{X}/2$ HAS HALF THE MAGNITUDE OF $\vec{X}$, AND THE DIRECTION IS FLIPPED 180° FROM THE DIRECTION OF $\vec{X}$. BOTH VECTORS ARE STILL ON THE SAME LINE OF ACTION.

2- & 3-DIMENSIONAL VECTORS

WE WILL DESCRIBE 2- & 3-DIMENSIONAL VECTORS USING CARTESIAN COORDINATES. IN TWO DIMENSIONS, THE CARTESIAN COORDINATE SYSTEM HAS TWO PERPENDICULAR COORDINATE AXES, TYPICALLY LABELED x AND y.



POINT P HAS COORDINATES (a, b)



VECTOR $\vec{F}$ HAS COMPONENT MAGNITUDES F_x AND F_y

THE MAGNITUDE OF A 2-DIMENSIONAL VECTOR CAN BE FOUND USING THE PYTHAGOREAN THEOREM:

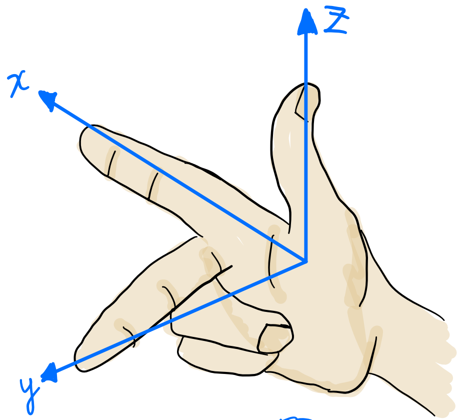
$$|\vec{F}| = \sqrt{F_x^2 + F_y^2}$$

THE DIRECTION θ OF A 2-DIMENSIONAL VECTOR CAN BE FOUND USING THE TANGENT OF θ:

$$\theta = \tan^{-1}\left(F_y / F_x\right)$$

WE CAN EASILY EXTEND THE 2D COORDINATE SYSTEM TO 3D BY ADDING A z-AXIS, PERPENDICULAR TO BOTH THE x- AND y-AXES.

ITS CONVENIENT TO ASSIGN A CONVENTION FOR THE ORIENTATION OF THE x-, y-, AND z- AXES. TO DO THIS, WE USE THE RIGHT-HAND RULE.



USING THE FIRST THREE FINGERS OF YOUR RIGHT HAND AS PICTURED, YOU CAN DETERMINE THE APPROPRIATE ORIENTATION OF A RIGHT-HANDED COORDINATE SYSTEM.

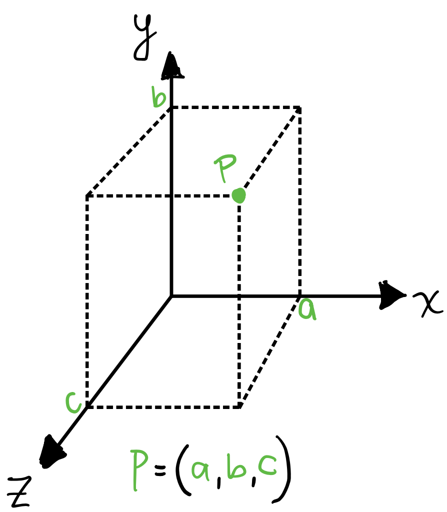
YOU CAN TWIST YOUR HAND TO LINE UP THE COORDIATE SYSTEM WITH ANY GIVEN POINTS, VECTORS, ETC.

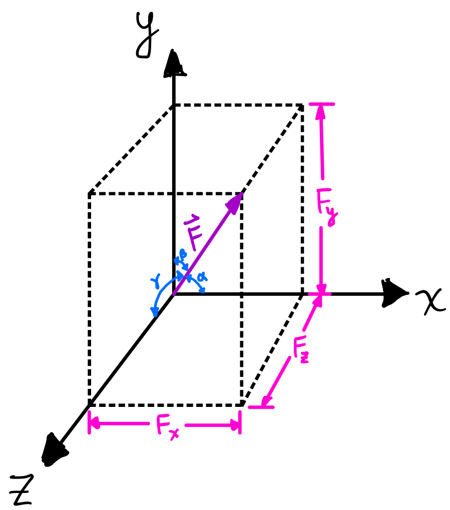
YOU CAN ALSO RELABEL THE AXES, AS LONG AS THE ORDER STAYS THE SAME.

PICTURED

THUMB —	x	y	z
INDEX —	y	z	x
MIDDLE —	z	x	y

POINTS AND VECTORS ARE EXPRESSED SIMILARLY AS IN 2D, WITH A THIRD COMPONENT REPRESENTING THE MAGNITUDE/DISTANCE ALONG THE Z-AXIS.





$\vec{F} = \langle F_x, F_y, F_z \rangle$

THE MAGNITUDE OF 3D VECTOR $\vec{F}$ IS:

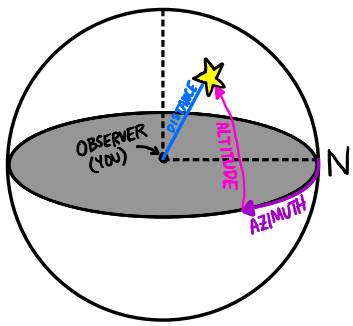
$$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

THE COORDINATE DIRECTION ANGLES ARE THE ANGLES THAT $\vec{F}$ MAKES WITH EACH OF THE COORDINATE AXES. THEY CAN BE FOUND USING COSINES:

$$\cos\alpha = \frac{F_x}{|\vec{F}|} \qquad \cos\beta = \frac{F_y}{|\vec{F}|} \qquad \cos\gamma = \frac{F_z}{|\vec{F}|}$$

NOTE THAT F_x, F_y AND F_z CAN BE NEGATIVE, BUT $|\vec{F}|$ WILL ALWAYS BE POSITIVE.

SOMETIMES, IT WILL BE MORE CONVENIENT TO DESCRIBE A POINT OR VECTOR USING SPHERICAL COORDINATES. FOR EXAMPLE, IMAGINE YOU ARE STARGAZING AND WANT TO DESCRIBE THE LOCATION OF POLARIS (THE NORTH STAR).



DISTANCE: $A \cong 433$ LIGHT YEARS

AZIMUTH: $\theta = 1°$

ALTITUDE: $\phi = 30°$

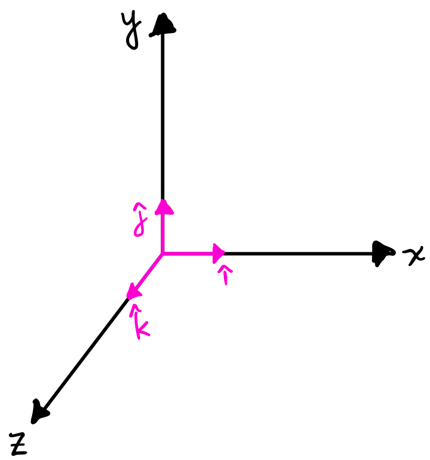
$\Big\}$ (CAN VARY ±1° OVER TIME)

WE CAN CONVERT ANY VECTOR $\vec{A}$ FROM SPHERICAL TO RECTANGULAR COORDINATES AND VICE VERSA USING TRIGONOMETRY.

$\vec{A} = \langle A_x, A_y, A_z \rangle$	$\vec{A} = (A; \theta; \phi)$
$A = \lvert\vec{A}\rvert = \sqrt{A_x^2 + A_y^2 + A_z^2}$	$A_x = A\sin\phi\cos\theta$
$\phi = \cos^{-1}\left(\frac{A_z}{A}\right)$	$A_y = A\sin\phi\cos\theta$
$\theta = \cos^{-1}\left(\frac{A_x}{A\sin\phi}\right)$	$A_z = A\cos\phi$

UNIT VECTORS

A UNIT VECTOR IS A VECTOR WITH A MAGNITUDE OF 1 AND NO UNITS. BY CONVENTION, A UNIT VECTOR IS INDICATED BY A HAT OVER THE VECTOR INSTEAD OF AN ARROW. THE UNIT VECTORS THAT POINT ALONG THE x-, y-, AND z-AXES OF A CARTESIAN COORDINATE SYSTEM ARE SO COMMON, THEY'VE BEEN GIVEN THEIR OWN SYMBOLS : $\hat{\imath}$, $\hat{\jmath}$, AND $\hat{k}$



WITH UNIT VECTORS, WE CAN EXPRESS A VECTOR $\vec{F}$ WITH COMPONENTS F_x, F_y AND F_z AS:

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

THE UNIT VECTOR $\hat{F}$ POINTS IN THE SAME DIRECTION AS $\vec{F}$, BUT HAS A MAGNITUDE OF 1.

$$\hat{F} = \frac{\vec{F}}{|\vec{F}|} = \frac{F_x}{|\vec{F}|}\hat{\imath} + \frac{F_y}{|\vec{F}|}\hat{\jmath} + \frac{F_z}{|\vec{F}|}\hat{k}$$

THIS LOOKS VERY SIMILAR TO OUR EXPRESSIONS FOR THE COORDINATE DIRECTION ANGLES. IN FACT, WE CAN WRITE

$$\hat{F} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$$

REMEMBER THAT MULTIPLYING OR DIVIDING A VECTOR BY A SCALAR CHANGES THE VECTOR'S MAGNITUDE, BUT RETAINS THE ORIGINAL LINE OF ACTION.

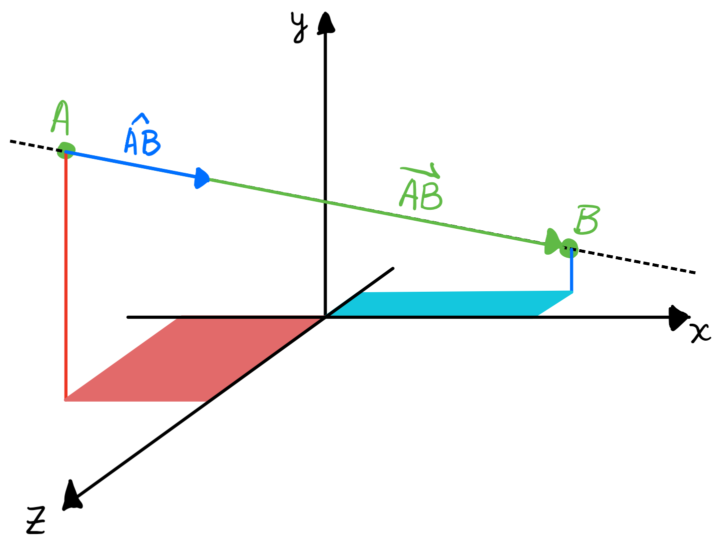
MAGNITUDE & DIRECTION → $\vec{F} = \underbrace{|\vec{F}|}_{\text{MAGNITUDE}}\,\underbrace{\hat{F}}_{\text{DIRECTION}}$

ONE WAY TO THINK ABOUT UNIT VECTORS →

- VECTOR $\vec{F}$ HAS MAGNITUDE & DIRECTION
- SCALAR $|\vec{F}|$ HAS ONLY MAGNITUDE
- UNIT VECTOR $\hat{F}$ HAS ONLY DIRECTION

UNIT VECTORS ARE TYPICALLY THE BEST WAY TO DEAL WITH FORCES AND DISTANCES IN THREE DIMENSIONS.

FOR EXAMPLE, IF WE KNOW THE LOCATION OF TWO POINTS (A & B) ON THE LINE OF ACTION OF FORCE $\vec{F}$, THEN WE CAN USE A UNIT VECTOR TO DETERMINE THE COMPONENTS OF $\vec{F}$. HERE'S HOW:



$A = (A_x, A_y, A_z)$

$B = (B_x, B_y, B_z)$

0. DRAW A GOOD DIAGRAM! (AS SHOWN ABOVE.)

1. USE THE KNOWN POINTS TO FIND THE DISPLACEMENT VECTOR $\vec{AB}$:

$$\vec{AB} = (B_x - A_x)\hat{\imath} + (B_y - A_y)\hat{\jmath} + (B_z - A_z)\hat{k}$$

OR, WRITE OUT THE DISPLACEMENTS DIRECTLY:

$$AB_x = \Delta x = B_x - A_x$$
$$AB_y = \Delta y = B_y - A_y$$
$$AB_z = \Delta z = B_z - A_z$$

$$\vec{AB} = AB_x\hat{\imath} + AB_y\hat{\jmath} + AB_z\hat{k}$$

2. FIND THE DISTANCE BETWEEN POINTS A AND B. THIS IS ALSO THE MAGNITUDE OF THE DISPLACEMENT VECTOR $\vec{AB}$:

$$|\vec{AB}| = \sqrt{(AB_x)^2 + (AB_y)^2 + (AB_z)^2}$$

3. FIND THE UNIT VECTOR $\hat{AB}$. THIS IS A UNITLESS VECTOR WITH A MAGNITUDE OF 1 THAT POINTS FROM A TO B.

$$\hat{AB} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{AB_x}{|\vec{AB}|}\hat{\imath} + \frac{AB_y}{|\vec{AB}|}\hat{\jmath} + \frac{AB_z}{|\vec{AB}|}\hat{k}$$

4. FINALLY, MULTIPLY THE MAGNITUDE OF THE FORCE (F_{AB}) BY THE UNIT VECTOR $\hat{AB}$ TO GET THE FORCE VECTOR $\vec{F}_{AB}$

$$\vec{F}_{AB} = F_{AB}\,\hat{AB}$$

TIPS FOR WORKING WITH UNIT VECTORS:

- THE SIGNS OF THE UNIT VECTOR COMPONENTS SHOULD MATCH THE SIGNS OF THE RESPECTIVE COMPONENTS OF THE ORIGINAL VECTOR.
- COMPONENTS OF A UNIT VECTOR MUST BE BETWEEN −1 AND +1.